Special Topics in Continuous Optimization and Optimal Control

Local and Global convergence of Newton methods

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Overview

1. Introduction
2. Background
3. Local Conv.
4. Failures: NM
5. Theorems
6. Global Conv.
7. Globalization Schemes
8. Conclusion

Introduction

Problem:

- Finding x^* such that $F(x^*) = 0$
- $F : \mathbb{R}^n \to \mathbb{R}^n$ (highly) nonlinear
- Important problem in continuous optimization and optimal control

Newton's method:

- Iterative method
- Solve linearized problem
- Many variants

Globalization strategies:

- Newton's method only locally convergent
- Hope to globalize convergence

Lipschitz Condition

- Given g: [a,b] $\rightarrow \mathbb{R}$ is called Lipschitz continuous with constant $\lambda > 0$ (denoted g ϵ Lip_λ[a,b]) if $\exists\lambda > 0$ such that $|g(x) - g(y)| \leq \lambda|x - y|$ for all x,y ϵ [a,b]

Contraction map

- g: [a,b] $\rightarrow \mathbb{R}$ is called contraction map if g ϵ Lip_λ[a,b] with $\lambda < 1$

Convex set

- A set C is convex if, for any x,y ϵ C and θ ϵ $\mathbb{R}$ with $0 \leq \theta \leq 1$, $\theta x + (1 - \theta)y\epsilon$ C

Convex function

- A function f: $\mathbb{R} \rightarrow \mathbb{R}$ is convex if its domain (denoted D(f)) is a convex set and if, for all x,y ϵ D(f) and θ ϵ $\mathbb{R}$ with $0 \leq \theta \leq 1$,
- $f(\theta x + (1 - \theta)y) \leq \theta f(x) + (1 - \theta)f(y)$

Summary: Newton Method

- Fast (i.e. quadratic) local rate of convergence
- Scale-invariant w.r.t linear transformations of the variables
- Search direction p^k is not well defined if $\nabla^2 \mathrm{f}(\mathrm{x}^\mathrm{k})$ is singular, p^k is not a descent if $\nabla^2 \mathrm{f}(\mathrm{x}^\mathrm{k})$ is not positive definite
- Minimum points x^k can be attracted to sadle points or local maxima of f
- Very small neighbourhood of local convergence, Newton's method is not globally convergent
- Line search, trust region

1. Convergence properties of Newton methods

What is convergence?

- Convergence means **approaching a limit** as the argument of the function increases or decreases or as the number of terms in the series increases.
- Types of convergence: local and global
- **When does it converge locally?** When the initial approximation is already **close enough to the solution**, then the succesive aproximations of the iterative method guranteed to converge to a solution locally.
- Iterative methods for nonlinear equations and their systems, such as Newton's method are usually only locally convergent.

Rate of convergence

- $\{x^k\} \subset \mathbb{R}^n, x^* \epsilon \mathbb{R}^n, \{x^k\} \to x^*$ as k $\to \infty$
- $\{x^k\} \to x^k$ with rate r if $\frac{\|x^{k+1}-x^*\|}{\|x^k-x^*\|^r} = c < \infty$, for sufficiently large k
- r=1: linear convergence (c < 1)
- r=2: quadratic convergence
- superlinear convergence: $\frac{\|x^{k+1}-x^*\|}{\|x^k-x^*\|^r} \to 0$ as k $\to \infty$

Local Convergence Theorem

- Single dimension: If
- $\frac{\partial f}{\partial x}$ bounded away from zero
- $\frac{\partial^2 f}{\partial x^2}$ bounded
- Then Newton's method converges given a sufficiently close initial guess (and convergence is quadratic)

Multidimensional

- If $\|J_F^{-1}(x^k)\| \leq \beta$ (inverse is bounded)
- $\|J_F(x) - J_F(y)\| \leq l\|x - y\|$ (Derivative is Lipschitz continuous)

Example 1

$$f(x) = x^2 - 1 = 0, \quad find \ \ x \ \ (x^* = 1)$$

$$\frac{df}{dx}(x^k) = 2x^k$$

$$2x^k(x^{k+1} - x^k) = -\left(\left(x^k\right)^2 - 1\right)$$

$$2x^k(x^{k+1} - x^*) + 2x^k(x^* - x^k) = -\left(\left(x^k\right)^2 - \left(x^*\right)^2\right)$$

$$or \ \ (x^{k+1} - x^*) = \frac{1}{2x^k}(x^k - x^*)^2$$

- We see that the convergence is quadratic

Example 2

$$f(x) = x^2 = 0, \quad x^* = 0$$

$$\frac{df}{dx}(x^k) = 2x^k$$

$$\Rightarrow \; 2x^k(x^{k+1} - 0) = (x^k - 0)^2$$

$$x^{k+1} - 0 = \frac{1}{2}\left(x^k - 0\right) \qquad \text{for } x^k \neq x^* = 0$$

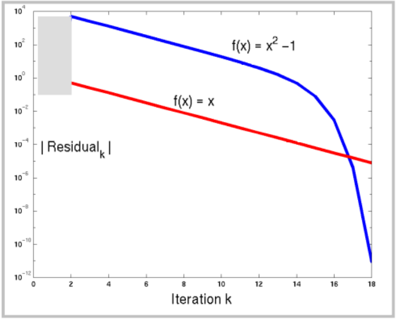
$$or \; (x_{k+1} - x^*) = \frac{1}{2}(x_k - x^*)$$

- Note: $\frac{\partial f}{\partial x}^{-1}$ not bounded away from zero
- We see the convergence is linear

Plot

Newton-Raphson Method – Convergence

Example 1,2



December 6, 2019 <number> courtesy Alessandra Nardi UCB

Convergence algorithm

Newton-Raphson Method – Convergence

$x^0 =$ Initial Guess, $k = 0$

Repeat {

$$\frac{\partial f(x^k)}{\partial x}(x^{k+1} - x^k) = -f(x^k)$$

$$k = k + 1$$

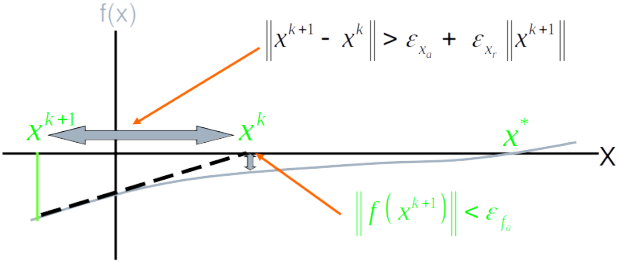
} Until ?

$\left\|x^{k+1} - x^k\right\| < threshold$? $\left\|f(x^{k+1})\right\| < threshold$?

Newton-Raphson Method – Convergence

Convergence Check

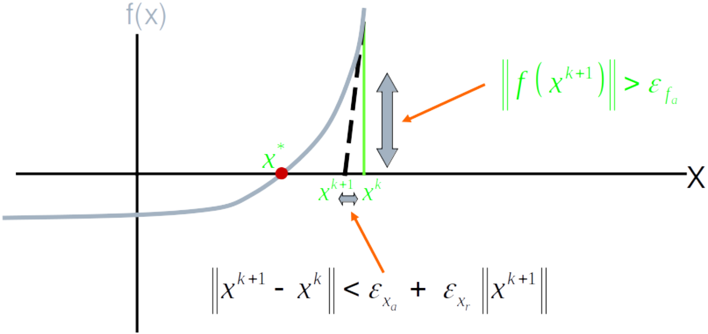
Need a "delta-x" check to avoid false convergence



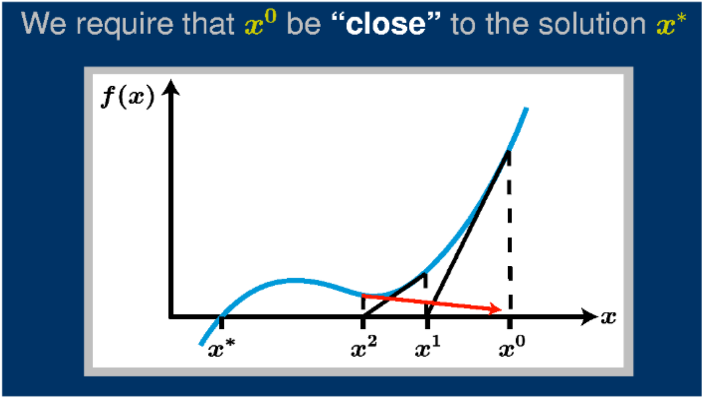
Newton-Raphson Method – Convergence

Convergence Check

Also need an "$f(x)$" check to avoid false convergence



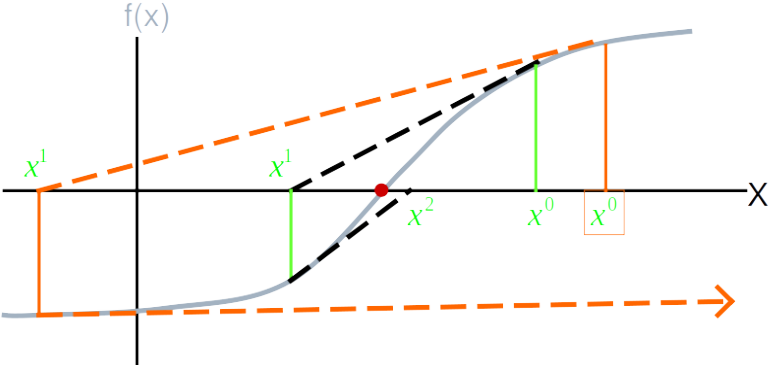
Newton-Raphson Method – Convergence



Newton-Raphson Method – Convergence

Local Convergence

Convergence Depends on a Good Initial Guess



Newton-Raphson Method – Convergence

Local Convergence

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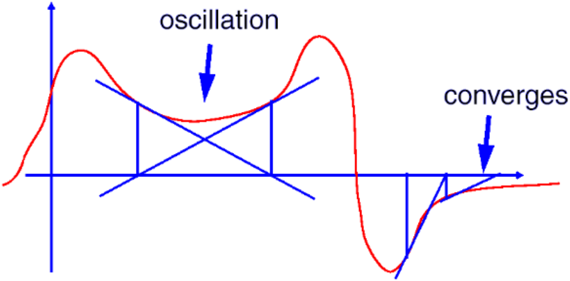
Example:



Advantages of Newton Method:

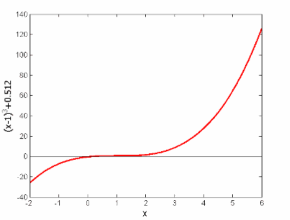
- Quadratic converges
- Requires only one guess
- This is very fast if we are close to a solution
- Doubles the correct digits in each iteration!

Drawbacks of Newton methods:

- **Divergence at inflection points**: If the initial guess or an iteration value of the root that is close to the inflection point of the function.

Table 1 Divergence near inflection point.

Iteration Number	x_i
0	5.0000
1	3.6560
2	2.7465
3	2.1084
4	1.6000
5	0.92589
6	−30.119
7	−19.746
18	0.2000



Divergence at inflection point for $f(x) = (x-1)^3 + 0.512 = 0$

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Figure: Divergence at inflection points

Drawbacks: Division by zero

- $f(x) = x^3 - 0.03x^2 + 2.4 \times 10^{-6} = 0$

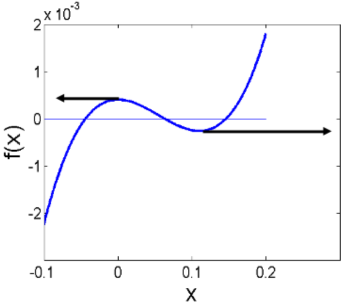

Figure: Pitfall of division by zero or near a zero number

- For $x_0 = 0 or x_0 = 0.02$ the denominator will be zero

Drawbacks: Oscillations

- Oscillations near local maxima and minimum
- Results may oscillate abut the local maximum or minimum without converging on a root but converging on the local maximum or minimum.
- Leads to division by a number close to zero and may diverge.
- $f(x) = x^2 + 2 = 0$ has no real roots

Table 2 Oscillations near local maxima and mimima

Iteration Number	x_i	$f(x_i)$	$\lvert\in_a\rvert\%$
0	–1.0000	3.00	
1	0.5	2.25	300.00
2	–1.75	5.063	128.571
3	–0.30357	2.092	476.47
4	3.1423	11.874	109.66
5	1.2529	3.570	150.80
6	–0.17166	2.029	829.88
7	5.7395	34.942	102.99
8	2.6955	9.266	112.93
9	0.97678	2.954	175.96

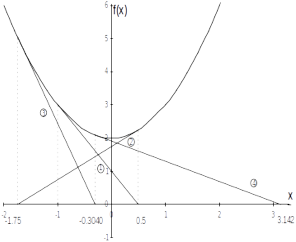

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Figure: Oscillations around local minima for $f(x) = x^2 + 2$

Drawbacks: Root jumping

- In some cases, where the function f(x) is oscillating and has a number of roots, one may choose an initial guess close to a root.
- However, the guesses may jump and converge to some other root.

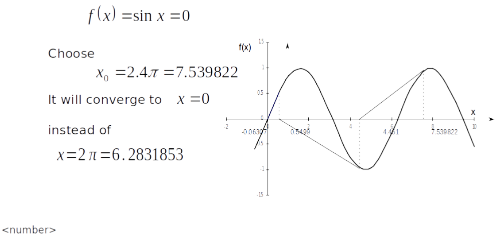


Figure: Root jumping from intended location of root for f(x)=$\sin$ x

Theorem

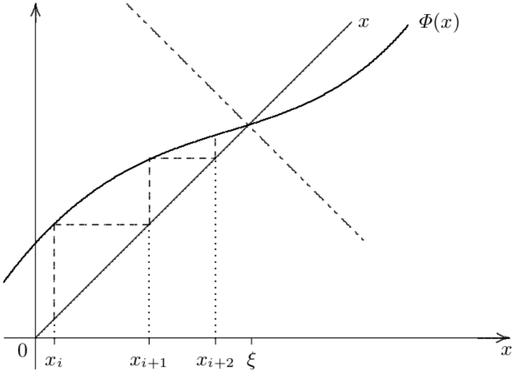
- Theorem. Let $\Phi : \mathbb{R}^n \to \mathbb{R}^n$ be an iterative function with fixed point ξ. U(ξ) is a neighbourhood of ξ, a number p $\geq$ 1 and a constant C $\geq$ 0 (with C $\leq$ 1 if p = 1) so that for all x ϵ U(ξ)

$$\|\Phi(x) - \xi\| \leq C\|x - \xi\|^p \tag{1}$$

- Then there is a neighbourhood (subset) V(ξ) $\subset$ U(ξ) of ξ so that for all starting points x_0 the iteration method defined by Φ geneerates the iteration steps x_i ϵ V(ξ) $\forall$ i $\geq$ 0 that converges to ξ at least with order p

Example

- E=$\mathbb{R}$, Φ is differentiable in a neighbourhood U(ξ) If $0 < \Phi'(\xi) < 1$, then the convergence will be linear (first order), x_i will converge monotonically to ξ



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General Convergence Theorem

- Let the function Φ: E $\rightarrow$ E, E= $\mathbb{R}^n$, have a fixed point ξ: Φ $(\xi) = \xi$
- Further let $S_r(\xi)$:={ z ||| z - ξ|| < r} be a neighbourhood of ξ such that Φ is a contractive mapping in S_r (ξ) that is

$$\|\Phi(x) - \Phi(y)\| \leq K\|x - y\| \quad (2)$$

0 $\leq$ K < 1 for all x,y ϵ $S_r(\xi)$. Then for any x_0 ϵ $S_r(\xi)$, the generated sequence $x_{i+1} = \Phi(x_i)$, i =0,1,.., has the following properties

- $x_i \epsilon$ $S_r(\xi)$ $\forall$ i=0,1,..,
- $\|x_{i+1} - \xi\| \leq K\|x_i - \xi\| \leq K^{i+1}\|x_0 - \xi\|$ i.e., $\{x_i\}$ converges at least linearly to ξ

Banach Fixed Point Theorem

- Let $\Phi : E \to E$, $E = \mathbb{R}^n$ be an iterative fucnction, $x_0 \epsilon E$ be a starting point, and $x_{i+1} = \Phi(x_i)$, i=0,1, ... Further,let a neighbourhood $S_r(x_0) = \{x | \|x - x_0\| <$ r$\}$ of x_0 and a constant K where 0 < K < 1, exist such that
  - $\|\Phi(x) - \Phi(y)\| \leq K\|x - y\|$ for all x,y ϵ $\overline{S_r(x_0)} := \{x | \|x - x_0\| \leq r\}$
  - $\|x_1 - x_0\| = \|\Phi(x_0) - x_0\| \leq$ (1 - K)r < r

  Then it follows that
  - $x_i \ \epsilon \ S_r(x_0) \forall$ i = 0, 1, ...,
  - Φ has exactly one fixed point ξ, $\Phi(\xi) = \xi$, in $\overline{S_r(x_0)}$ and $\lim_{i\to\infty} x_i = \xi$,

    $\|x_{i+1} - \xi\| \leq K\|x_i - \xi\|$, as well as $\|x_1 - \xi\| \leq \frac{K^i}{1-K}\|x_1 - x_0\|$

Proof of quadratic convergence

Theorem. Assume that f is twice continuously differentiable on an open interval (a, b) and that there exists $x^* \in (a, b)$ with $f'(x^*) \neq 0$. Define Newton's method by the sequence

$$x_{k+1} = x_k - \frac{f(x_k)}{f'(x_k)}, \qquad k = 1, 2, \dots$$

Assume also that x_k converges to x^* as $k \to \infty$. Then, for k sufficiently large,

$$|x_{k+1} - x^*| \leq M|x_k - x^*|^2 \qquad \text{if } M > \frac{|f''(x^*)|}{2|f'(x^*)|}.$$

Thus, x_k converges to x^* *quadratically* (A&G, p. 52).

Proof. Let $e_k = x_k - x^*$, so that $x_k - e_k = x^*$. By Taylor's Theorem (A&G, Chap. 1, p. 5), setting $x = x_k$ and $h = -e_k$, we have

$$f(x_k - e_k) = f(x_k) - e_k f'(x_k) + \frac{(e_k)^2}{2} f''(\xi_k)$$

for some ξ_k between x_k and x^*. Since $x_k - e_k = x^*$ and $f(x^*) = 0$, we have

$$0 = f(x_k) - (x_k - x^*) f'(x_k) + \frac{(e_k)^2}{2} f''(\xi_k).$$

Since the derivative of f is continuous with $f'(x^*) \neq 0$, we have $f'(x_k) \neq 0$ as long as x_k is close enough to x^*. So we can divide by $f'(x_k)$ to give

$$0 = \frac{f(x_k)}{f'(x_k)} - (x_k - x^*) + \frac{(e_k)^2 f''(\xi_k)}{2f'(x_k)},$$

which, by the definition of Newton's method, gives

$$x_{k+1} - x^* = \frac{(e_k)^2 f''(\xi_k)}{2f'(x_k)}.$$

So

$$|x_{k+1} - x^*| \leq \frac{|f''(\xi_k)|}{2|f'(x_k)|} |x_k - x^*|^2.$$

By continuity, $f'(x_k)$ converges to $f'(x^*)$ and, since ξ_k is between x_k and x^*, ξ_k converges to x^* and hence $f''(\xi_k)$ converges to $f''(x^*)$, so, for large enough k,

$$|x_{k+1} - x^*| \leq M|x_k - x^*|^2 \qquad \text{if } M > \frac{|f''(x^*)|}{2|f'(x^*)|}.$$

In fact, it can be shown without assuming that x_k converges to x^*, that there exists $\delta > 0$ such that, if $|x_0 - x^*| \leq \delta$, then x_k converges to x^*, and hence from the above argument that the convergence rate is quadratic, but this requires a more complicated argument by induction.

1. Minimization Problems
2. Global Convergence Definition
3. Globalization Schemes
4. Conclusions

Minimization Problems

We consider the following minimization problem for a real function $h : \mathbb{R}^n \to \mathbb{R}$ of n variables

$$\min_x h(x).$$

Definition of Global Convergence

- The iterative method is called locally convergent with $V(\overline{x})$, a neigborhood of $\overline{x}$, if it generates, for all starting points $x_0 \in V(\overline{x})$, a sequence $\{x_k\}$ that converges to $\overline{x}$.

[Note that $\overline{x}$ is a minimum point for the function h.]

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- It is called globally convergent, if in addition $V(\overline{x}) = \mathbb{R}^n$.

[Note that $\overline{x}$ is a minimum point for the function h.]

Important Lemma

- Let $h : \mathbb{R}^n \to \mathbb{R}$ be a function which has a continuous derivative $Dh(x)$ for all $x \in V(\overline{x})$.

[Note that $D(\gamma, x) := \{s \in \mathbb{R}^n | \|s\| = 1, Dh(x)s \geq \gamma \|Dh(x)\|\}$ and $Dh(x) = \nabla h(x)^T = (\frac{\partial h(x)}{\partial x^1}, \cdots, \frac{\partial h(x)}{\partial x^n})$.]

Important Lemma

- Let $h : \mathbb{R}^n \to \mathbb{R}$ be a function which has a continuous derivative $Dh(x)$ for all $x \in V(\overline{x})$.
- Suppose further that $Dh(\overline{x}) \neq 0$, and let $1 \geq \gamma > 0$.

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- Suppose further that $Dh(\overline{x}) \neq 0$, and let $1 \geq \gamma > 0$.

Then there is a neighborhood $U(\overline{x}) \subseteq V(\overline{x})$ of $\overline{x}$ and a number $\lambda > 0$ such that

$$h(x - \mu s) \leq h(x) - \frac{\mu\gamma}{4} \|Dh(\overline{x})\|$$

for all $x \in U(\overline{x}), s \in D(\gamma, x)$ and $0 \leq \mu \leq \lambda$.
[Note that $D(\gamma, x) := \{s \in \mathbb{R}^n | \|s\| = 1, Dh(x)s \geq \gamma \|Dh(x)\|\}$ and $Dh(x) = \nabla h(x)^T = \left(\frac{\partial h(x)}{\partial x^1}, \cdots, \frac{\partial h(x)}{\partial x^n} \right)$.]

Globalization Schemes

- Modified Newton Method with Exact Line Search

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- Modified Newton Method with Exact Line Search
- Modified Newton Method with Inexact Line Search
- Quasi-Newton Method

Globalization Schemes

- Modified Newton Method with Exact Line Search
- Modified Newton Method with Inexact Line Search
- Quasi-Newton Method
  - BFGS
  - Oren-Luenberger

Modified Newton Method with Exact Line Search

- Choose a starting point $x_0 \in \mathbb{R}^n$.
  Choose numbers $\gamma_k \leq 1, \sigma_k, k = 0, 1, .., $ with $\inf_k \gamma_k > 0, \inf_k \sigma_k > 0.$

Modified Newton Method with Exact Line Search

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  Choose numbers $\gamma_k \leq 1, \sigma_k, k = 0, 1, .., $ with $\inf_k \gamma_k > 0, \inf_k \sigma_k > 0$.
- For all $k = 0, 1, ...,$ update

$$x_{k+1} := x_k - \lambda_k s_k$$

where $s_k \in D(\gamma_k, x_k)$, and $\lambda_k \in [0, \sigma_k \|Dh(x_k)\|]$ is such that

$$h(x_{k+1}) = \min_{\mu}\{h(x_k - \mu s_k) : 0 \leq \mu \leq \sigma_k \|Dh(x_k)\|\}.$$

Global Convergence Results for Modified Newton Method with Exact Line Search

- $K := \{x | h(x) \leq h(x_0)\}$ is compact, and
- h is continuously differentiable in some open set containing K.

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- $K := \{x|h(x) \leq h(x_0)\}$ is compact, and
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Then for any sequence $\{x_k\}$ defined by this method:

- $x_k \in K$ for all $k = 0, 1, ...$ The sequence $\{x_k\}$ has at least one accumulation point $\overline{x}$ in K.
- Each accumulation point $\overline{x}$ of $\{x_k\}$ is a stationary point of h:

$$Dh(\overline{x}) = 0.$$

Proof for the Method with Exact Line Search: 1

From the definition of the sequence $\{x_k\}$ we have that the sequence $\{h(x_k)\}$ is monotone, i.e., $h(x_0) \geq h(x_1) \geq \cdots$. Hence, $x_k \in K$ for all k. K is compact; therefore, the sequence $\{x_k\}$ has at least one accumulation point $\overline{x} \in K$.

Proof for the Method with Exact Line Search: 2-(1)

Assume that $\overline{x}$ is an accumulation point of $\{x_k\}$ but is not a stationary point of h:

$$Dh(\overline{x}) \neq 0. \tag{3}$$

WLOG, let $\lim_{k\to\infty} x_k = \overline{x}$.
According to the important lemma, there is a neighborhood $U(\overline{x})$ and a number $\lambda \geq 0$ satisfying

$$h(x - \mu s) \leq h(x) - \mu \frac{\gamma}{4} \|Dh(\overline{x})\| \tag{4}$$

for all $x \in U(\overline{x}), s \in D(\gamma, x)$, and $0 \leq \mu \leq \lambda$.
Since $\lim_{k\to\infty} x_k = \overline{x}$, the continuity of $Dh(x)$ together with (3) implies there is k_0 such that for all $k \geq k_0, x_k \in U(\overline{x})$ and $\|Dh(x_k)\| \geq \frac{1}{2}\|Dh(\overline{x})\|$.

Proof for the Method with Exact Line Search: 2-(2)

Let $\Lambda := \min\{\lambda, \frac{1}{2}\sigma\|Dh(\overline{x})\|, \epsilon := \Lambda\frac{\gamma}{4}\|Dh(\overline{x})\|\}$.
Since $\Lambda \leq \lambda, x_k \in U(\overline{x}), s_k \in D(\gamma_k, x_k)$, (4) implies that
$h(x_{k+1}) \leq h(x_k) - \Lambda\frac{\gamma}{4}\|Dh(\overline{x})\| = h(x_k) - \epsilon$ for all $k \geq k_0$.
This means that $\lim_{k\to\infty} h(x_k) = -\infty$, which contradicts the fact that $h(x_k) \geq h(\overline{x})$ for all k. Hence, $\overline{x}$ is a stationary point of h.

Modified Newton Method with Inexact Line Search

- Choose a starting point $x_0 \in \mathbb{R}^n$.
  Choose numbers $\gamma_k \leq 1, \sigma_k, k = 0, 1, .., $ with $\inf_k \gamma_k > 0, \inf_k \sigma_k > 0$.
- For all $k = 0, 1, ...,$ obtain x_{k+1} from x_k as follows:
  - Select $s_k \in D(\gamma_k, x_k)$.
    Define $\rho_k := \sigma_k \|Dh(x_k)\|$, $h_k(\mu) := h(x_k - \mu s_k)$.
    Then, determine the smallest integer $j \geq 0$ such that
    $h_k(\rho_k 2^{-j}) \leq h_k(0) - \rho_k 2^{-j} \frac{\gamma_k}{4} \|Dh(x_k)\|$.
  - Determine $\bar{i} \in \{0, 1, ..., j\}$ such that $h_k(\rho_k 2^{-\bar{i}})$ is minimum and let
    $\lambda_k := \rho_k 2^{-\bar{i}}$.
    Then, update $x_{k+1} := x_k - \lambda_k s_k$.

[Note that $h(x_{k+1}) = \min_{1 \leq i \leq j} h_k(\rho_k 2^{-i})$.]

Global Convergence Results for Modified Newton Method with Inexact Line Search

- $K := \{x|h(x) \leq h(x_0)\}$ is compact, and
- h is continuously differentiable in some open set containing K.

Then for any sequence $\{x_k\}$ defined by this method:

- $x_k \in K$ for all $k = 0, 1, \ldots$. The sequence $\{x_k\}$ has at least one accumulation point $\overline{x}$ in K.
- Each accumulation point $\overline{x}$ of $\{x_k\}$ is a stationary point of h:

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From the definition of the sequence $\{x_k\}$ we have that the sequence $\{h(x_k)\}$ is monotone, i.e., $h(x_0) \geq h(x_1) \geq \cdots$. Hence, $x_k \in K$ for all k. K is compact; therefore, the sequence $\{x_k\}$ has at least one accumulation point $\overline{x} \in K$.

Proof for the Method with Inexact Line Search: 2

Again, we will prove the second result by a contradiction, which is similar to the previous proof in the section of exact line seach. Assume that $\overline{x}$ is an accumulation point of a sequence $\{x_k\}$ but not a stationary point of h, i.e.,

$$Dh(\overline{x}) \neq 0.$$

By the important lemma and the hypotheses of the global convergence results, we can show that there is an $\epsilon \geq 0$ for which

$$h(x_{k+1}) \leq h(x_k) - \epsilon$$

for all $k > k_0$. This contradicts the fact that $h(x_k) \geq h(\overline{x})$ for all k. Therefore, $\overline{x}$ is a stationary point of h.

Quasi-Newton Methods

- Choose a starting point $x_0 \in \mathbb{R}^n$ and an $n \times n$ positive definite matrix H_0. Set $g_0 := g(x_0)$.
- For $k = 0, 1, \dots$ obtain x_{k+1}, H_{k+1} from x_k, H_k as follows:
  - if $g_k = 0$, stop: x_k is a stationary point for h. Otherwise
  - compute $s_k := H_k g_k (\approx H(x_k)^{-1} g_k)$.
  - Update $x_{k+1} = x_k - \lambda_k s_k$ by means of a minimization

$$h(x_{k+1}) \approx \min\{h(x_k - \lambda s_k) | \lambda \geq 0\},$$

$$g_{k+1} := g(x_{k+1}), p_k := x_{k+1} - x_k, q_k := g_{k+1} - g_k.$$

  - Choose suitable parameters $\gamma_k > 0, \theta_k \geq 0$, and compute $H_{k+1} = \psi(\theta_k, \gamma_k H_k, p_k, q_k)$ where

$$\psi(\theta, H, p, q) := H + (1 + \theta \frac{q^T H q}{p^T q}) \frac{p p^T}{p^T q}$$

$$- \frac{(1-\theta)}{q^T H q} H q \cdot q^T H - \frac{\theta}{p^T q} (p q^T H + H q p^T).$$

Global Convergence Results for Quasi-Newton Method (BFGS)

- $H(\overline{x})$ is positive definite.

[Note that $H(x) := \left(\frac{\partial^2 h(x)}{\partial x^i \partial x^k} \right)_{i,k=1,\ldots,n}$.]

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Global Convergence Results for Quasi-Newton Method (BFGS)

- $H(\overline{x})$ is positive definite.
- $H(x)$ is Lipschitz continuous at $x = \overline{x}$.
- Given constants $0 < c_1 < c_2 < 1, c_1 \leq \frac{1}{2}$, $x_{k+1} = x_k - \lambda_k s_k$ is chosen so that

$$h(x_{k+1}) \leq h(x_k) - c_1 \lambda_k g_k^T s_k,$$

$$g_{k+1}^T s_k \leq c_2 g_k^T s_k.$$

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Powell (1975) was able to show that $\exists V(\overline{x}) \subseteq U(\overline{x})$ such that the BFGS method is superlinearly convergent for all positive definite matrices H_0 and all $x_0 \in V(\overline{x})$. [Note that $H(x) := \left(\frac{\partial^2 h(x)}{\partial x^i \partial x^k} \right)_{i,k=1,\ldots,n}$.]

Global Convergence Results for Quasi-Newton Method (A Subclass of Oren-Luenberger)

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It can be shown Stoer (1977) that for all $k \geq 0$

$$\lim_k x_k = \overline{x}$$

$$\|x_{k+n} - \overline{x}\| \leq \gamma\|x_k - \overline{x}\|^2$$

for all positive definite initial matrices H_0 and for $\|x_0 - \overline{x}\|$ small enough

Conclusions for Local- and Global- Properties of Newton Methods

- If a starting point x_0 is chosen sufficiently close to the optimum point $\overline{x}$, under certain assumptions the sequence $\{x_n\}$ generated by Newton's method is at least locally convergent.

Conclusions for Local- and Global- Properties of Newton Methods

- If a starting point x_0 is chosen sufficiently close to the optimum point $\overline{x}$, under certain assumptions the sequence $\{x_n\}$ generated by Newton's method is at least locally convergent.
- Under certain conditions, the global convergence of Newton's method can be obtained by finding parameters λ_k and search directions s_k for the following update, $x_{k+1} = x_k - \lambda_k s_k$.

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Thank you very much for your attention.

Do you have any question?